

Chaos as a Limit in a Boundary Value Problem

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A piecewise-linear, 3-variable autonomous O.D.E. of C^0 type, known to describe constant-shape travelling waves in one-dimensional reaction-diffusion media of Rinzel-Keller type, is numerically shown to possess a chaotic attractor in state space. An analytical method proving the possibility of chaos is outlined and a set of parameters yielding Shil'nikov chaos indicated. A symbolic dynamics technique can be used to show how the limiting chaos dominates the behavior even of the finite boundary value problem.

Reaction-diffusion equations occur in many disciplines [1, 2]. Piecewise-linear systems are especially amenable to analysis. A variant to the Rinzel-Keller equation of nerve conduction [3] can be written as

$$\begin{aligned}\frac{\partial}{\partial t} u &= \frac{\partial^2}{\partial x^2} u + \mu[-u + v - \beta + \theta(u - \delta)], \\ \frac{\partial}{\partial t} v &= -\varepsilon u + v,\end{aligned}\quad (1)$$

where $\theta(a) = 1$ if $a > 0$ and zero otherwise; δ is the threshold parameter.

Focusing on those solutions to (1), which are wave-trains of constant shape, one arrives at the following 3-variable ordinary differential equation (see [4, 5]):

$$\begin{aligned}u' &= w, \\ v' &= \frac{\varepsilon u - v}{c}, \\ w' &= \mu[u - v + \beta - \theta(u - \delta)] - c w.\end{aligned}\quad (2)$$

Here $' = d/ds$ whereby s represents the wave variable, $x - ct$.

As compared to (1), (2) contains an additional free parameter, the travelling speed c . It has to be chosen such that the corresponding solution of (2) becomes a solution of (1) – with the latter subjected to the specific boundary conditions assumed.

In an earlier note [5], (2) was shown to produce finite wave-trains of arbitrary finite pulse number (and pulse arrangement) if either cyclic or natural

boundary conditions are assumed. This result was obtained by an analytical matching method. The connection to chaos theory (Smale [6] basic sets) was not evident at the time. In the following, even the possibility of manifest chaos will be demonstrated.

In Fig. 1, a chaotic attractor of (2) is presented. The flow can be classified as an example of “screw-type-chaos” (cf. [7]). A second example of a chaotic attractor is shown in Fig. 2. A 1-D projection of a 2-D cross section through the chaotic flow in a separating plane is also given (Fig. 2c). One-dimensional maps with a shape of this type generate chaos (cf. [8]). The (mathematically alone existing) attractor could nevertheless be an embedded limit cycle of high periodicity that is numerically inaccessible [8]. The presence of a narrow second dimension in the real map does not affect these conclusions (cf. [9]).

Equation (2), being piecewise linear, is also amenable to analytical methods, however. In the technique of “Poincaré half maps” [10], each half system is analyzed separately. Each half map is defined, from the “entry region” in the separating plane $u = \delta$ of one half system, into the corresponding “exit region”. The two half maps “flush” (that is, the entry region of one is the other's exit region and vice versa) because (2) has unique trajectories. The resulting combined overall map (two coupled implicit algebraic equations [10]) is an admissible “ordinary” Poincaré map.

In the examples of chaos found so far in (2) – Figs. 1 and 2 –, either half system contains a saddle focus in state space whose 1-D manifold is unstable (contracting eigenvector) while the 2-D manifold is

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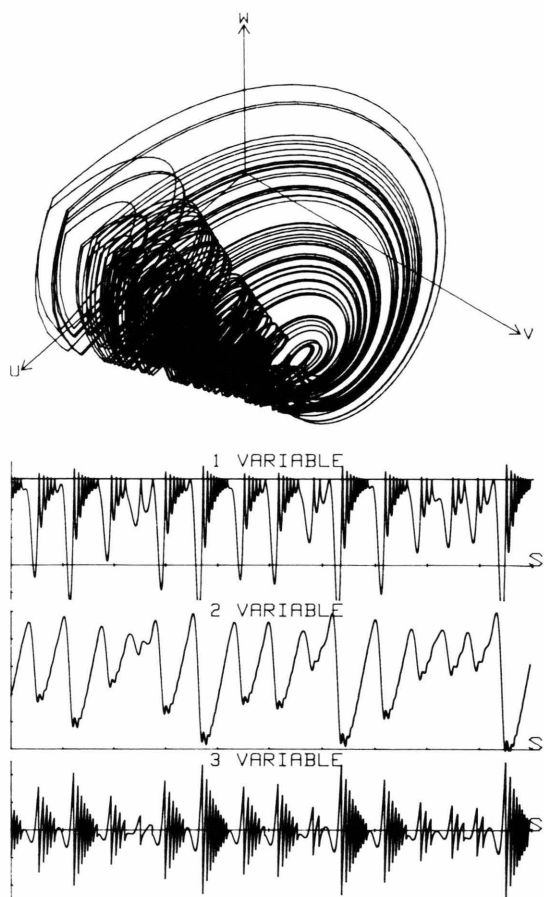


Fig. 1. Screw-type chaos in (2). Numerical solution of the initial value problem using a standard Runge-Kutta-Merson integration routine with step error 10^{-8} . Parameters chosen: $\beta = 1$, $\delta = 0.01$, $\varepsilon = 150$, $\mu = 10$, $c = 3.5$. Initial conditions: $u(0) = 0.009979$, $v(0) = 0.9285$, $w(0) = 0.06406$. Axes: -0.004 to 0.012 for u , 0.86 to 1.04 for v , -0.175 to 0.175 for w , 0 to 100 for s (upper trace), 0 to 30 (lower traces). The horizontal line in the first time plot indicates the threshold value $u = \delta$.

stable (containing the expanding focus). For the parameters chosen, only one of the two half systems possesses a “real” steady state of this type; the other has its steady state in the region of definition of the first one (“virtual” steady state) [5].

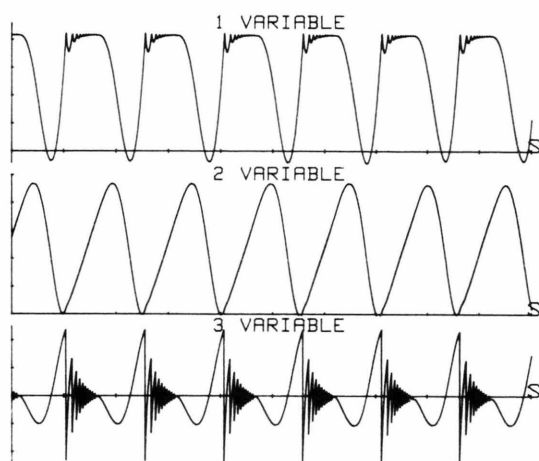
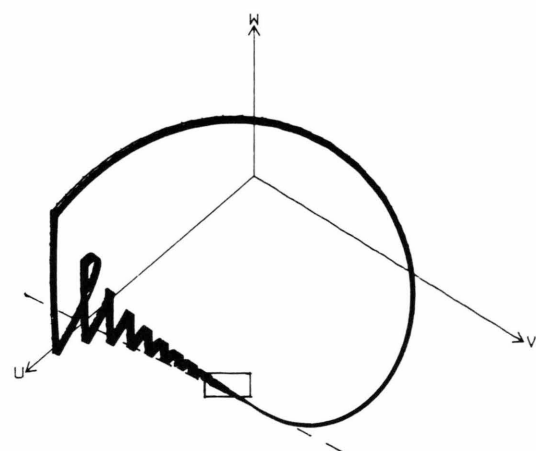
In the domain of the first half map one always finds a “distortion” (branch cut) that ends in a logarithmic spiral around a singular point [11]. This distortion generates a “splitting” in the half map [10]. The other half map in general does not undo this splitting. All one needs for chaos to occur is that

part of an image of a neighborhood of the distortion is mapped back into the same neighborhood. In that case the overall Poincaré map contains at least one Smale [6] horseshoe. Smale horseshoes, in turn, imply “nontrivial basic sets” [6] and hence infinitely many periodic trajectories of differing periods – implying chaos (non manifest at least). If both half maps are non-expanding (non-positive divergence in (2)), manifest chaos is possible [11].

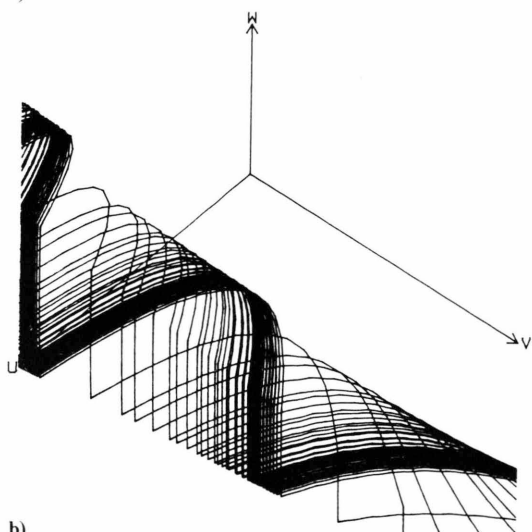
A special case within the above scenario is also analytically accessible. A trajectory that is “homoclinic” with respect to the saddle focus in state space is generated whenever the line of intersection of the two-dimensional manifold of the first half system with the separating plane is mapped back, by the second half map, in such a way that its image hits precisely the intersection point of the one-dimensional manifold of the first half system with the separating plane. Such homoclinicity to a saddle focus in state space was shown by Shil’nikov [12] to imply the formation of infinitely many Smale horseshoes (under a mild condition of the eigenvalues). One particular set of parameters for (2) yielding a situation that fulfills the conditions of Shil’nikov’s theorem is as follows: $\beta = 1$, $\delta = 0.01$, $\varepsilon = 150$, $\mu = 10$, $c = 4.30523810140 \dots$ (cf. [5]). Note that the Shil’nikov case corresponds to the innermost part of the spiral distortion (mentioned above) being mapped back upon itself.

Any such “ordinary” homoclinic trajectory corresponds to a single-pulse wave train on the infinite linear fiber in (1). Similarly, if not the first but the second image of the straight line mentioned above hits the one-dimensional manifold as described, the corresponding homoclinic situation implies a double-pulse wave-train, and so forth up to arbitrarily long finite wave trains (see the examples, up to order 10, in [5]). Cf. also [13], where similar results were predicted for the Hodgkin-Huxley equation.

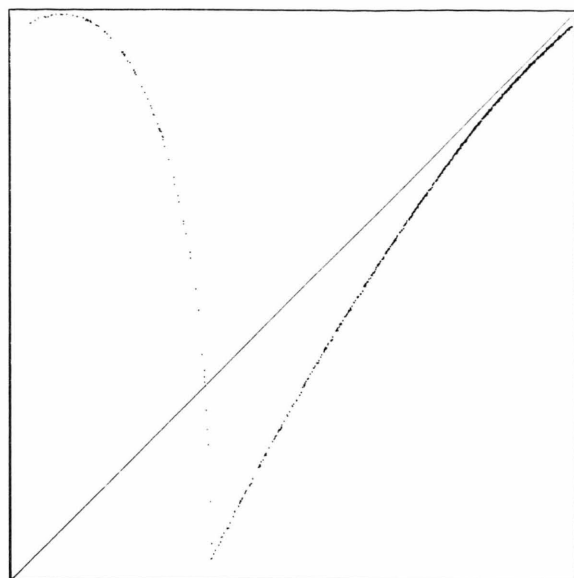
The connection between this finite wave-train case and the infinite case is nontrivial, however. Finite pulse sequences by definition cannot represent wave chaos. A symbolic dynamics method can be used to make the relationship more transparent. The excitable medium is first divided into “distance quanta” [5, 14]. Then, for any solution the presence or absence of a supra-threshold ($u_{\max} > \delta$) pulse in each bin can be symbolized by a “1” or “0”, respectively. The number N of realized com-



a)



b)



c)

Fig. 2. A second type of chaos in (2). Parameters: $\beta = 1$, $\delta = 0.01$, $\varepsilon = 150$, $\mu = 10$, $c = 9.0$. (a) Phase space plot for 100 s -units. Initial conditions: $u(0) = 0.01000$, $v(0) = 0.9985$, $w(0) = -0.003726$. Axes: -0.001 to 0.011 for u , 0.96 to 1.03 for v , -0.05 to 0.05 for w , 0 to 15 for s . (b) Blow up. Axes: 0.0099 to 0.01 for u , 1.003 to 1.008 for v , -0.0025 to 0.0025 for w . (c) Approximate 1-D map of the interval $(1.003, 1.008)$ of the dashed line in the $u = \delta$ plane indicated in (a). About 2200 points (up to $s = 5000$) are shown.

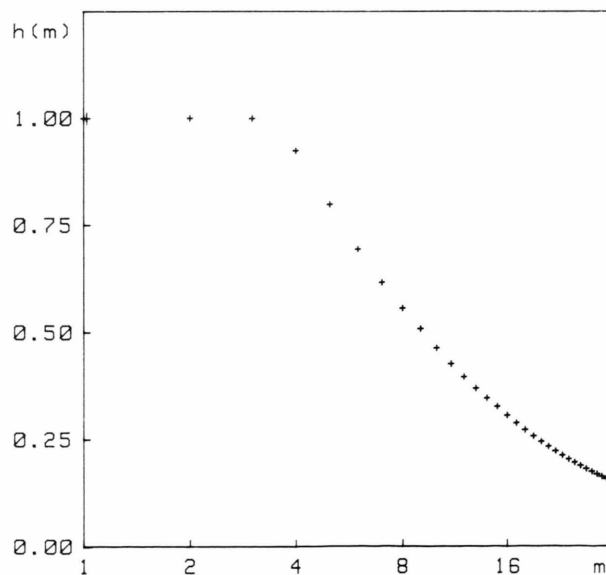


Fig. 3. Plot of $h(m)$ for the 10-pulse wave train indicated in the text.

binations of 0's and 1's as a function of the length, m , of a "running window", can be obtained for this solution.

In this way, a quantity h can be arrived at that is defined as follows:

$$h(m) := \frac{1}{m} \log_2 N(m). \quad (3)$$

As an example, take the wave train of 10 pulses as shown in Fig. 16c [5]. It possesses the following symbolic representation: "101111100110011", with zeros in front and behind. A plot of $h(m)$ for this sequence is shown in Figure 3.

The point is that if we had taken a longer wave train of comparable local complexity, we would have obtained essentially the same graph once more. Only the left-hand portion of the graph (with unit height in the present case) would be elongated. (For comparable-complexity wave-trains, this elongation is proportional to the logarithm of wave-train length.) The "tail" (which in the case of Fig. 3 starts at $m=4$) would have essentially the same form again, going asymptotically to zero due to the $1/m$ factor in (3).

Mathematically, the straight (or wiggly, in other cases) "plateau" in the left-hand portion of the graph of Fig. 3 has no name. However, if it had infinite length (because we had analyzed a complex infinite-length wave-train) then its value would be the topological entropy of the corresponding infinite wave-train and, by implication, of the corresponding chaotic regime (cf. [15]).

We think that the presence of this potentially growing plateau ("pre-entropy") can be used to differentiate between asymptotically chaotic and not asymptotically chaotic boundary value problems

within the present class — also and even if only finite lengths are accessible. This conjecture opens the possibility that even a finite boundary value problem may be shown to be "related to chaos".

Finally the stability problem is worth mentioning. A wave-train in (1), being described by a trajectory of (2), will in general not share the stability properties of the latter. Nevertheless for the chaotic attractor of Fig. 2, the original P.D.E. can be expected to possess an attracting set of chaotic waves. (Small perturbations of a wave-train would lead back to this set, but not necessarily to the same element.) This is because the wave velocity c in Fig. 2a ($c = 9.0$) is more than twice the value of that of a single unstable pulse ($c = 4.3 \dots$), for the same parameters, in (1). (See the above "Shil'nikov set".) Although it is known (cf. [4, 5]) that multiple unstable pulses may travel somewhat faster than single ones, a speed increase by more than 100% appears highly unlikely. Therefore, the wave train described indirectly by Fig. 2 probably consists of stable pulses. It hereby goes without saying that such "temporal stability" of the individual components of a wave train does not necessarily imply "spatial stability" of the wave train [16].

To sum up, wave-train chaos exists in 1-D excitable media. The infinite wave-train case is embedded in an exploding scenario of possibilities in which the limiting case (chaos) makes its presence felt even in the finite boundary value problem already.

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